Hedge Strategies and Timings with Non-linear Transaction Cost Functions *

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Abstract

This paper introduces a new model optimizing the hedge strategy with non-linear transaction costs. Though transaction costs were assumed to be proportional to the transaction volume so far, in the new model I assumed transaction costs are discounted as the volume increases and it was closer to real markets. The model shows that the hedge interval is inversely proportional to the 4th square of volatility and the square of gamma. This implies that increases in volatility or gamma accelerates the frequency of the transactions, and is informative in analyzing the market liquidity, where volatility and gamma increase.

1 Introduction

Until now, most of the research of hedge strategy model with transaction costs are under the limited conditions: (a) dealers hedge only European call option; (b) dealers change the hedge portfolio with constant time interval; and (c) transaction costs are proportional to the amount of transactions. Both of the famous models of Leland (1985) and Boyle-Vorst (1992) also were based on such conditions, and proposed the improvements of the delta hedge strategy in the Black-Scholes model.

On the other hand, van der Hoek-Platen (1996) assumed that it is possible to make up the hedge error (the difference between values of the contingent claims and hedge portfolios) with added funds. And they aimed to decide the hedge strategy to control the expected transaction costs and variance (risk) of added funds in the future to be as small as possible. This model is useful because it can be applied under the looser conditions: this model (a) can be used for not only European call options but also all European contingent claims; (b) optimizes the changeable timings of transactions; (c) argues with not only proportional but also constant transaction costs. However, they didn't solve the problem clearly, and I believe that it is relevant that transaction costs are discounted as volume increases.

This paper aims at the extended model of van der Hoek-Platen, and has the following characteristics: the model (a) is also applied with non-linear transaction costs; (b) clearly solves the most suitable strategy that the expected transaction costs are the smallest under the circumstance that the variance (risk) of added funds doesn't exceed the limit each dealer preset; (c) examines the form of the transaction cost function which maximizes the broker's commission income.

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2 Risk Control Hedge Strategy Model

2.1 The Model

In the same way as the Black-Scholes model, let's consider the market with one risky asset and one risk-free bond with constant unit value. Let the stochastic process $S = \{S_t;\ 0 \le t \le T\}$ with $0 < T < \infty$ describe the price of the risky asset as the solution of the Ito stochastic differential equation

$$dS_t = \sigma S_t dW_t, \quad 0 \le t \le T, \tag{1}$$

where $W = \{W_t, 0 \le t \le T\}$ is a given standard Wiener process and $\sigma > 0$ is a constant volatility. [1]

Within this paper, let's consider the problem of hedging a given contingent claim of the form

$$H = g(S_T) \tag{2}$$

on the price of risky asset S_T at the maturity T in the presence of transaction costs. The no-arbitrage price without transaction costs is expressed by the form

$$u(t, S_t) = E(g(S_T)|\mathcal{F}_t), \tag{3}$$

where u is a smooth function on $[0,T) \times \mathbb{R}^+$, that is infinitely often differentiable in $t \in [0,T)$ and $S_t \in \mathbb{R}^+$.

Transactions mean changes in the allocation of the portfolio occurring according to a finite time discretization $t = \tau_0, \tau_1, \dots, \tau_{i_T}$

$$0 = \tau_0 < \tau_1 < \cdots < \tau_{i_T}, \quad i_t = \max\{i = 0, 1, \dots;\ \tau_i \le t\}. \tag{4}$$

Let's assume that any time discretization $\tau_0, \tau_1, \dots, \tau_{i_T}$ with given $\Delta > 0$ is characterized by a strictly positive $\mathcal{F}-$ predictable scaling process $\delta = \{\delta_t;\ 0 \le t \le T\}$ such that

$$\tau_{i_t+1} = \min\{t | t \ge \tau_{i_t} + \frac{\Delta}{\delta_t}\} \tag{5}$$

for all $t \in [0, T]$.

A dynamical hedge strategy $\phi = \{\phi_t = (\xi_t, \eta_t, \delta_t)\ ;\ 0 \le t \le T\}$ is a strategy to build a portfolio at time t, including ξ_t sheres of risky asset and η_t units of the bond. Let's assume that the costs of transactions with the bond are nothing and those with the risky asset are

$$\Lambda_{\tau_i}(\phi) = \lambda \Delta^q \{S_{\tau_i} |\xi_{\tau_i} - \xi_{\tau_{i-1}}|\}^l, \tag{6}$$

where $\lambda > 0$, $0 \le l \le 1$, and q is the fixed positive number which satisfies

$$\lim_{\Delta \to 0} \sum_{i=0}^{i_T} \Lambda_{\tau_i}(\phi) = 0, \tag{7}$$

for all mean-self-financing strategy ϕ which is defined in the following. The dependency on Δ reflects that transaction costs have to be lower as liquidity of securities is higher. The equation (7) implies that the transaction costs of securities which you can deal continually are 0.

[1] These conditions imply that both of the risk-free interest rate r and expected earning rate μ are 0 in the Black-Scholes model. $r = 0$ means that we consider all securities prices with the value measured in the money at time 0, and $\mu = 0$ means that probability measure of the standard Wiener process W is the risk neutral measure from the first. Therefore, These conditions are essentially the same ones as the Black-Scholes model.

Given a dynamical hedge ϕ, let's define its value process

$$V_t(\phi) = \xi_t S_t + \eta_t, \tag{8}$$

which represents the value of the hedge portfolio at time $t \in [0, T]$.

Furthermore, let's introduce for a given hedge strategy ϕ its cost process

$$C_t(\phi) = V_t(\phi) - \int_0^t \xi_s dS_s + \sum_{i=0}^{i_t} \Lambda_{\mathcal{T}_i}(\phi) \tag{9}$$

being the value of the portfolio without gain from trade but including the cumulative transaction costs up to time $t \in [0, T]$. Let's also define for given hedge strategy ϕ the expected transaction costs

$$U_{t,v}(\phi) = E\Big(\sum_{i=i_t+1}^{i_v} \Lambda_{\mathcal{T}_i}(\phi)|\mathcal{F}_t\Big), \quad t \in [0, T], v \in [t, T]. \tag{10}$$

Let's say that a trading strategy ϕ hedge against H if

$$V_T(\phi) = H. \tag{11}$$

A dynamical trading strategy ϕ will be called admissible if it hedges against H,

$$E\Big(\int_0^T |\xi_t|^2 \sigma^2 S_t^2 dt|\mathcal{F}_0\Big) < \infty, \quad E\Big(\big(V_t(\phi) + \sum_{i=0}^{i_t} \Lambda_{\mathcal{T}_i}(\phi)\big)^2|\mathcal{F}_0\Big) < \infty, \quad t \in [0, T], \tag{12}$$

$\eta = \{\eta_t;\ 0 \le t \le T\}$ is $\mathcal{F}$-adapted, and $\xi = \{\xi_t;\ 0 \le t \le T\}$ is piecewise constant with smooth function on $\mathcal{T}_{i_t}$ and $S_{\mathcal{T}_{i_t}}$. Let's say that an admissible strategy ϕ is mean-self-financing if

$$E(C_T(\phi) - C_t(\phi)|\mathcal{F}_t) = 0 \tag{13}$$

for all $t \in [0, T]$.

Let's introduce for a given mean-self-financing strategy ϕ the notion of risk at time t to v

$$R_{t,v}(\phi) = E((C_v(\phi) - C_t(\phi))^2|\mathcal{F}_t) \tag{14}$$

which represents the conditional variance of the costs under $\mathcal{F}_t$.

2.2 Asymptotically Risk Hedge Strategy

Given a mean-self-financing strategy $\phi = \{\phi_t = (\xi_t, \eta_t, \delta_t);\ 0 \le t \le T\}$, according to (9), (13),

$$V_t(\phi) = E\Big(V_v(\phi) + \sum_{i=i_t+1}^{i_v} \Lambda_{\mathcal{T}_i}(\phi)|\mathcal{F}_t\Big) \tag{15}$$

for all $t \in [0, T]$, $v \in [t, T]$. When $v = T$, according to (11), (2), (3), (10),

$$V_t(\phi) = u(t, S_t) + U_{t,T}(\phi), \tag{16}$$

which means that the value of portfolio is the sum of no-arbitrage price without transaction costs and the expected transaction costs which will be necessary by maturity T.

Let's note that any given random variable $Y \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ can be decomposed for $t \in [0, T]$ in the form

$$Y = E(Y|\mathcal{F}_t) + \int_t^T \mu_s(Y) dS_s \tag{17}$$

(see Kunita-Watanabe (1967)), where $\mu(Y) = \{\mu_s(Y), t \leq s \leq T\}$ is an $\mathcal{F}_T$-predictable process, and linear function which satisfies $E(\int_t^T \mu_s^2(Y)ds|\mathcal{F}_t) < \infty$.

According to (14), (9), (15), and (1),

$$\begin{aligned} R_{t,v}(\phi) &= E(\int_t^v (\mu_s(V_v(\phi) + \sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi)) - \xi_s)dS_s)^2|\mathcal{F}_t) \\ &= \int_t^v E((\mu_s(V_v(\phi) + \sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi)) - \xi_{\tau_{i_s}})^2\sigma^2 S_s^2|\mathcal{F}_t)ds. \end{aligned} \tag{18}$$

According to Kunita-Watanabe and (15), for all $v \in [t, T]$,

$$V_T(\phi) + \sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi) = V_t(\phi) + \int_t^T \mu_s(V_T(\phi) + \sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi))dS_s, \tag{19}$$

$$V_v(\phi) + \sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi) = V_t(\phi) + \int_t^v \mu_s(V_v(\phi) + \sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi))dS_s. \tag{20}$$

And from Kunita-Watanabe, (15) and linear characteristics of $\mu(\cdot)$

$$V_T(\phi) + \sum_{i=i_v+1}^{i_T} \Lambda_{\tau_i}(\phi) = V_v(\phi) + \int_v^T \mu_s(V_T(\phi) + \sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi))dS_s - \int_v^T \mu_s(\sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi))dS_s. \tag{21}$$

Moreover, from Kunita-Watanabe,

$$\int_v^T \mu_s(\sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi))dS_s = \sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi) - E(\sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi)|\mathcal{F}_v) = 0. \tag{22}$$

And subtracting (20) and (21) from (19),

$$0 = \int_t^v (\mu_s(V_T(\phi) + \sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)) - \mu_s(V_v(\phi) + \sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi)))dS_s \tag{23}$$

This is true for all $t \in [0, v]$, and according to the linear characteristics of $\mu(\cdot)$ and (11),

$$\mu_s(V_v(\phi) + \sum_{i=i_t+1}^{i_v} \Lambda_{\tau_i}(\phi)) = \mu_s(H) + \mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)), \quad s \in (0, v). \tag{24}$$

And so according to (18),

$$R_{t,v}(\phi) = \int_t^v E((\mu_s(H) + \mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)) - \xi_{\tau_{i_s}})^2\sigma^2 S_s^2|\mathcal{F}_t)ds. \tag{25}$$

And for all $v \in [\tau_{i_t+1}, T]$,

$$\begin{aligned} R_{\tau_{i_t},v}(\phi, \Delta) &= \xi_{\tau_{i_t}}^2 \int_{\tau_{i_t}}^{\tau_{i_t+1}} E(\sigma^2 S_s^2|\mathcal{F}_{\tau_{i_t}})ds - 2\xi_{\tau_{i_t}} \int_{\tau_{i_t}}^{\tau_{i_t+1}} E((\mu_s(H) + \mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)))\sigma^2 S_s^2|\mathcal{F}_{\tau_{i_t}})ds \\ &\quad + \int_{\tau_{i_t}}^{\tau_{i_t+1}} E((\mu_s(H) + \mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)))^2\sigma^2 S_s^2|\mathcal{F}_{\tau_{i_t}})ds \\ &\quad + \int_{\tau_{i_t+1}}^{v} E((\mu_s(H) + \mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)) - \xi_{\tau_{i_s}})^2\sigma^2 S_s^2|\mathcal{F}_{\tau_{i_t}})ds. \end{aligned} \tag{26}$$

And so the risk minimizing strategy at $\tau_{i_t} \in [0,T)$ is expressed with

$$\xi_{\tau_{i_t}} = \frac{\int_{\tau_{i_t}}^{\tau_{i_t+1}} E((\mu_s(H) + \mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)))\sigma^2 S_s^2 | \mathcal{F}_{\tau_{i_t}})ds}{\int_{\tau_{i_t}}^{\tau_{i_t+1}} E(\sigma^2 S_s^2 | \mathcal{F}_{\tau_{i_t}})ds}. \tag{27}$$

When Δ is small enough,

$$\xi_t = \mu_t(H) + \mu_t(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)). \tag{28}$$

From (25) and (28),

$$R_{t,v}(\phi) = \int_t^v E((\mu_s(H) - \mu_{\tau_{i_s}}(H) + \mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)) - \mu_{\tau_{i_s}}(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)))^2 \sigma^2 S_s^2 | \mathcal{F}_t)ds. \tag{29}$$

Otherwise, according to (7) and linear characteristics of $\mu(\cdot)$,

$$\lim_{\Delta \to 0} \mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)) = 0. \tag{30}$$

With $\Delta \to 0$, from (28),

$$\xi_t = \mu_t(H). \tag{31}$$

And let's call the strategy with (31) asymptotically risk minimizing strategy. Now according to Kunita-Watanabe and (3), for all $t \in [0,T]$,

$$H = u(t, S_t) + \int_t^T \mu_s(H)dS_s, \tag{32}$$

and let's regard this as a function of t and S_t, and apply the Ito formula to obtain

$$0 = (\frac{\partial}{\partial S_t} u(t, S_t) - \mu_t(H))dS_t \tag{33}$$

for all $t \in [0,T]$. In this calculation, we use the following equation which is Black-Scholes's differential equation with $r = 0$,

$$\mathcal{L}_t^0(u(t, S_t)) = 0, \quad t \in [0,T] \quad (\mathcal{L}_t^0 = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2}). \tag{34}$$

And so according to (31) and (33)

$$\xi_t = \mu_t(H) = \frac{\partial}{\partial S_t} u(t, S_t). \tag{35}$$

This is consistent with the Delta Hedge Strategy.

Now according to (35) and (30),

$$\begin{aligned}
\lim_{\Delta \to 0} \frac{\mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)) - \mu_{\tau_{i_s}}(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi))}{\mu_s(H) - \mu_{\tau_{i_s}}(H)} &= \lim_{\Delta \to 0} \frac{\frac{\partial}{\partial s}\mu_s(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi))}{\frac{\partial}{\partial s}\mu_s(H)} \\
&= \frac{\frac{\partial}{\partial s}(\lim_{\Delta \to 0} \mu_{\tau_{i_s}}(\sum_{i=i_t+1}^{i_T} \Lambda_{\tau_i}(\phi)))}{\frac{\partial^2}{\partial s \partial S_s} u(s, S_s)} \\
&= 0.
\end{aligned} \tag{36}$$

And with asymptotically risk minimizing strategy, according to (35) and (34)

$$\mathcal{L}_t^0(\mu_t(H)) = 0, \tag{37}$$

and so according to Ito's formula and (1),

$$\mu_t(H) = \mu_{\tau_{i_t}}(H) + \int_{\tau_{i_t}}^{t} \sigma S_s \frac{\partial^2}{\partial S_s^2} u(s, S_s) dW_s \tag{38}$$

is true for all $t \in [0, T]$. Hence with the continuity of S_t, $\frac{\partial^2}{\partial S_t^2} u(t, S_t)$, and δ_t, from (29), the following equation is obtained.

$$\begin{aligned}
\lim_{\Delta \to 0} \frac{R_{t,v}(\phi)}{\Delta} &= \lim_{\Delta \to 0} \frac{1}{\Delta} \sum_{i=i_t+1}^{i_v} \int_{\tau_{i-1}}^{\tau_i} E((\int_{\tau_{i-1}}^{s} \sigma S_z \frac{\partial^2}{\partial S_z^2} u(z, S_z) dW_z)^2 \sigma^2 S_s^2 | \mathcal{F}_t) ds \\
&= \lim_{\Delta \to 0} \sum_{i=i_t+1}^{i_v} E(\frac{1}{\Delta} \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^{s} (\sigma S_z \frac{\partial^2}{\partial S_z^2} u(z, S_z))^2 \sigma^2 S_s^2 dz ds | \mathcal{F}_t) \\
&= \lim_{\Delta \to 0} \sum_{i=i_t+1}^{i_v} E(\frac{1}{\Delta} (\sigma^2 S_{\tau_i}^2 \frac{\partial^2}{\partial S_{\tau_i}^2} u(\tau_i, S_{\tau_i}))^2 \frac{1}{2} (\frac{\Delta}{\delta_{\tau_i}})^2 | \mathcal{F}_t) \\
&= \frac{1}{2} \int_t^v E((\sigma^2 S_s^2 \frac{\partial^2}{\partial S_s^2} u(s, S_s))^2 \delta_s^{-1} | \mathcal{F}_t) ds
\end{aligned} \tag{39}$$

Based on above all, when Δ is small enough, the following equation is proved.

$$R_{t,v}(\phi) = \frac{\Delta}{2} \int_t^v E((\sigma^2 S_s^2 \frac{\partial^2}{\partial S_s^2} u(s, S_s))^2 \delta_s^{-1} | \mathcal{F}_t) ds. \tag{40}$$

Next, let's obtain the expected transaction cost with asymptotically risk minimizing strategy.

$$E(|W_{\tau_i} - W_{\tau_{i-1}}|^l | \mathcal{F}_{\tau_{i-1}}) = \frac{2^{\frac{l}{2}}}{\sqrt{\pi}} (\frac{\Delta}{\delta_{\tau_i}})^{\frac{l}{2}} \Gamma(\frac{l+1}{2}), \tag{41}$$

and so according to (10), (6), (35), (38), with the continuity used above,

$$\begin{aligned}
\lim_{\Delta \to 0} \Delta^{1-\frac{l}{2}-q} U_{t,v}(\phi) &= \lim_{\Delta \to 0} E(\sum_{i=i_t+1}^{i_v} \lambda \{S_{\tau_i} | \int_{\tau_{i-1}}^{\tau_i} \sigma S_s \frac{\partial^2}{\partial S_s^2} u(s, S_s) dW_s | \}^l \Delta^{1-\frac{l}{2}} | \mathcal{F}_t) \\
&= \lim_{\Delta \to 0} E(\sum_{i=i_t+1}^{i_v} \lambda (\{S_{\tau_{i-1}} \sigma S_{\tau_{i-1}} | \frac{\partial^2}{\partial S_{\tau_{i-1}}^2} u(\tau_{i-1}, S_{\tau_{i-1}}) || W_{\tau_i} - W_{\tau_{i-1}} | \}^l \Delta^{1-\frac{l}{2}} | \mathcal{F}_t) \\
&= \lambda \frac{2^{\frac{l}{2}}}{\sqrt{\pi}} \Gamma(\frac{l+1}{2}) \lim_{\Delta \to 0} E(\sum_{i=i_t+1}^{i_T} \frac{\Delta}{\delta_{\tau_i}} \sigma^l S_{\tau_{i-1}}^{2l} | \frac{\partial^2}{\partial S_{\tau_{i-1}}^2} u(\tau_{i-1}, S_{\tau_{i-1}}) |^l {\delta_{\tau_i}}^{1-\frac{l}{2}} | \mathcal{F}_t) \\
&= \lambda \frac{2^{\frac{l}{2}}}{\sqrt{\pi}} \Gamma(\frac{l+1}{2}) \int_t^T E(\sigma^l S_s^{2l} | \frac{\partial^2}{\partial S_s^2} u(s, S_s) |^l \delta_s^{1-\frac{l}{2}} | \mathcal{F}_t) ds.
\end{aligned} \tag{42}$$

Therefore when Δ is enough small, the following equation is obtained.

$$U_{t,v}(\phi) = \lambda \Delta^{q+\frac{l}{2}-1} \frac{2^{\frac{l}{2}}}{\sqrt{\pi}} \Gamma(\frac{l+1}{2}) \int_t^v E(\sigma^l S_s^{2l} | \frac{\partial^2}{\partial S_s^2} u(s, S_s) |^l \delta_s^{1-\frac{l}{2}} | \mathcal{F}_t) ds. \tag{43}$$

2.3 Risk Control Hedge Strategy

Now when Δ is enough small, let's consider the optimal asymptotically risk minimizing strategy with which the risk (40) at t does not exceed the upper limit $\kappa(t, v, S_t)$ each dealer set for all $v \in [t, T]$, that is

$$\frac{\Delta}{2}\int_t^v E((\sigma^2 S_s^2 \frac{\partial^2}{\partial S_s^2}u(s, S_s))^2 \delta_s^{-1}|\mathcal{F}_t)ds \leq \kappa(t, v, S_t). \tag{44}$$

With non-linear transaction cost functions in which the costs per volume discount as volume increases, it is better to hedge in long interval as long as the risk of being unable to hedge is under their risk aversion, and deal as large volume as possible. Therefore dealers do not have to deal as the risk does not reach the limit, and it is the best for them to use asymptotically risk minimizing strategy at the moment of reaching the limit. Accordingly, it is the best strategy to hedge at the first time t satisfying

$$t \geq \tau_{i_t} + \frac{\Delta}{\delta_t}, \tag{45}$$

in which δ_t satisfies (44). Let's call this strategy Risk Control Hedge Strategy.

When $\kappa(t, v, S_t)$ is differential in $v \in (t, T)$, let's make (44) equation, differential in v, and $v \to t$ to obtain

$$\delta_t = \frac{(\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2}u(t, S_t))^2}{2\lim_{v\to t}\frac{\partial}{\partial v}\kappa(t, v, S_t)}\Delta. \tag{46}$$

Now, as example, let's make the upper limit of risk

$$\kappa(t, v, S_t) = \kappa_0(v - t). \tag{47}$$

This means that the risk (the variance of added funds) from t to v is proportional to the time $v - t$ like the variance of Winner process. And the constant number κ_0 means the risk allowance of a dealer who can take more risk as κ_0 is larger. From (46) and (47),

$$\frac{\Delta}{\delta_t} = \frac{2\kappa_0}{\sigma^4 S_t^4\{\frac{\partial^2}{\partial S_t^2}u(t, S_t)\}^2}, \tag{48}$$

and so the hedge interval is proportional to the risk allowance κ_0, and inversely proportional to the 4th square of volatility and the square of gamma $\frac{\partial^2}{\partial S_t^2}u(t, S_t)$. And the following equations can be obtained.

$$R_{t,v}(\phi) = \kappa_0(v - t), \tag{49}$$

$$U_{t,v}(\phi) = \frac{\lambda\Delta^q}{2\sqrt{\pi\kappa_0}}\int_t^v E(\sigma^4 S_s^4|\frac{\partial^2}{\partial S_s^2}u(s, S_s)|^2|\mathcal{F}_t)ds\left(\frac{2\sqrt{\kappa_0}}{\sigma}\right)^l\Gamma(\frac{l+1}{2}) \tag{50}$$

3 Conclusion

In this paper, Risk Control Hedge Strategy model was introduced to hedge with non-linear transaction costs. This model clearly gave the most suitable strategy from the fact that it is better to hedge in long interval when transaction cost functions are non-linear. Moreover this model showed that the hedge interval should be proportional to the risk allowance (the upper limit of hedge risk differentiated by time), and inversely proportional to the 4th square of volatility and the square of gamma. This implies that increases in volatility or gamma accelerates the frequency of the transactions, and is informative in analyzing the market liquidity.

Risk Control Hedge Strategy was also shown to be effective, by the result of the numerical experiment with CRR model, to hedge the contingent claims whose payment in maturity was smooth with non-leaner transaction costs.

References

[1] Boyle, P.P. and Vorst, T. : "Option pricing in discrete time with transaction costs," *Journal of Finance* **47**, 271-293, 1992.

[2] Cox, J.C., Ross, S.A. and Rubinstein, M. : "Option pricing: a simplified approach," *Journal of Finance* **47**, 271-293, 1992.

[3] Harriaon, J.M. and Pliska, S.R. : "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications* **11**, 215-260, North-Holland Publishing Company, 1981.

[4] Ikeda, N. and Watanabe, S. : "*Stochastic Differential Equations and Diffusion Processes*(Second Edition)," North-Holland Mathematical Library, 1989.

[5] Leland, H.E. : "Option pricing and Replication with transaction costs," *Journal of Finance* **40**, 1283-1301, 1985.

[6] van der Hoek, J. and Platen, E. : "Pricing and Hedging in the Presence of Transaction Costs," *Working Paper,* 1996.

A Optimal Cost Function for Brokers

The transaction costs mean commission incomes for brokers. Now on the assumption that all dealers use Risk Control Hedge Strategy, let's consider the volume discount parameter l which maximizes the brokerage fee. according to (50), it is all right to consider l which maximizes $\left(\frac{2\sqrt{\kappa_0}}{\sigma}\right)^l \Gamma(\frac{l+1}{2})$. Then,

- $l = 1$, when $\kappa_0 > \frac{\pi\sigma^2}{4}$
- $l = 0$, when $\kappa_0 < \frac{\pi\sigma^2}{4}$.

This result means that when volatility is larger or dealers are more careful, they should hedge more frequently. Therefore brokers can earn more brokerage fee with constant transaction costs than linear ones. Otherwise when volatility is smaller or dealers are less careful, brokers should set linear transaction cost functions.