Which Risks Are Paid for with the Term Premium?♦

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February 2007

Abstract

Analytical solutions for term structures of nominal term premium, real term premium, and inflation risk premium, which are consistent with a new Keynesian model, are derived. Any of these premia are represented as a sum of three elements, each of which is a premium paid for risk of time-variation in each of IS shock, cost push shock, or monetary policy shock. The decomposition helps prove some counterintuitive properties of premia. For instance, a higher volatility of cost push shock or monetary policy shock, which is accompanied by a higher volatility of inflation, results in a lower inflation risk premium under plausible conditions.

JEL classification: E43; E52

Keywords: Term structure of interest rate, real rates of interest, new Keynesian model, term premium, inflation risk premium

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♦ This paper is a revised version of Chapter 3 of the author's doctoral dissertation submitted to UCSD in August, 2005. The author is deeply grateful to James D. Hamilton, for his valuable support and comments. The author also thanks Kosuke Aoki, Takeshi Kimura, Tuck Yun, and the seminar participants at UCSD, ECB, and the Bank of Japan for their helpful suggestions. All remaining errors are my own. The views expressed in this paper are the author's; they should not be attributed to the Bank of Japan.

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1. Introduction

The term structure of interest rates has been widely studied in the finance literature. The pioneer in this field is Vasicek (1977), which was followed by a large number of models. The literature succeeded in finding models with good empirical performance while keeping tractability, often obtaining closed form solutions even for complicated derivative security prices. However, as a caveat, many of the models have no macroeconomic interpretations. Typically, even the dynamics of the short rate are taken to be exogenous. On the other hand, the monetary economics literature has rapidly developed over the past decade, particularly in applications of the new Keynesian framework. This research employs dynamic general equilibrium models to find that the monetary policy rule—in which the short rate reacts to the output gap and the inflation rate, is crucial for determining the dynamics of the short rate and macroeconomic variables.

Recently, many studies have begun to try to integrate the finance and new Keynesian literatures for investigating the relationship between the interest rate term structure and macroeconomic variables. The pioneers of this kind of study are Piazzesi (2005) and Ang and Piazzesi (2003), both of which use reduced form models to examine the properties of the term structure when the short rate is determined by monetary policy rules. More recent studies such as Bekaert, Cho and Moreno (2005), and Wu (2006) use microfounded structural models. Most of these studies are mainly interested in explaining the time-variation of the term structure. For conducting this kind of study, they investigate factor loadings and impulse response functions; how much long-term interest rates depend on the factors and how the term structure responds to innovations.

This current paper is different from these studies in three aspects. First, this paper focuses on the average shape of term structure rather than the time-variation of that. A motivation comes from Figure 1, which displays average term structures of forward rates are robustly hump-shaped and peak at around maturities of 11-13 years. This suggests that the term structure of term premia of forward rates, which can be called the premium curve, is also hump-shaped on average. This fact is tried to be explained. Second, this paper examines not only nominal term premium but also real term premium and inflation risk premium. The properties of premia are investigated by decomposing the premia into three elements, each of which is interpreted as a premium paid for risk of time-variation in IS shock, cost push shock, or monetary policy shock. Finally, this paper uses a simple microfounded model. Although some ability to fit the data is sacrificed, the simple model makes it possible to obtain closed form solutions, which powerfully contribute to having clear intuition.

The subsequent sections are organized as follows. Section 2 reviews a simple new Keynesian model. Section 3 derives the analytical solution for nominal term premium, which is consistent with the new Keynesian model. The properties of the premium are then discussed. Section 4 considers real term premium and inflation risk premium. Section 5 concludes this paper.

2. A Simple New Keynesian Model

This section reviews a simple type of new Keynesian model with three variables, three shocks, and three equations. The dynamics of the variables, output gap x_t, inflation rate π_t, and short rate i_t, are determined by the three equations, the IS curve, the Phillips curve, and the monetary policy rule as

$$x_t = E_t x_{t+1} - \frac{1}{\gamma}(\hat{i}_t - E_t \hat{\pi}_{t+1} - \xi_t) \tag{1}$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa x_t + u_t \tag{2}$$

$$\hat{i}_t = \phi_x x_t + \phi_\pi \hat{\pi}_t + \omega_t \tag{3}$$

where $\hat{i}_t = i_t - \bar{i}$ and $\hat{\pi}_t = \pi_t - \bar{\pi}$ are defined as the deviations from the steady state values $\bar{i}$ and $\bar{\pi}$. The parameters satisfy $0 < \beta < 1$, $\gamma > 0$, $\kappa > 0$, $\phi_x > 0$ and $\phi_\pi > 1$. γ and β stand for the coefficient of relative risk aversion and the subjective discount factor of the representative household respectively, while κ is the slope of the Phillips curve, which reflects the degree of the nominal rigidity. The model has three exogenous shocks, which are assumed to obey AR(1) as

$$\xi_t = \rho_\xi \xi_{t-1} + \sigma_\xi \varepsilon_{\xi,t} \tag{4}$$

$$u_t = \rho_u u_{t-1} + \sigma_u \varepsilon_{u,t} \tag{5}$$

$$\omega_t = \rho_\omega \omega_{t-1} + \sigma_\omega \varepsilon_{\omega,t} \tag{6}$$

where $0 \le \rho_s < 1$, $\sigma_s > 0$ and $\varepsilon_{s,t} \sim N(0,1)$ i.i.d. for $s = \{\xi, u, \omega\}$. These shocks are called the IS shock, the cost push shock, and the monetary policy shock, respectively.

The IS curve (1) and the Phillips curve (2) are derived from microfoundations as shown in Appendix A. In particular, the microfoundation for the IS curve is important for

discussing the internal consistency between the IS curve and the interest rate term structure. The monetary policy rule (3) is the Taylor rule. This parsimonious model represents an advantage for obtaining clear intuition for the relationship between the term structure and macroeconomic variables.

The restrictions on the response coefficients in the monetary policy rule $\phi_x > 0$ and $\phi_\pi > 1$ are actually sufficient conditions for determining a unique stationary equilibrium,

$$\begin{pmatrix} x_t \\ \hat{\pi}_t \\ \hat{i}_t \end{pmatrix} = \begin{pmatrix} (1-\beta\rho_\xi)\Omega_\xi & -(\phi_\pi - \rho_u)\Omega_u & -(1-\beta\rho_\omega)\Omega_\omega \\ \kappa\Omega_\xi & \phi_x + \gamma(1-\rho_u)\Omega_u & -\kappa\Omega_\omega \\ b_\xi & b_u & b_\omega \end{pmatrix} \begin{pmatrix} \xi_t \\ u_t \\ \omega_t \end{pmatrix} \tag{7}$$

where the factor loadings of the short rate are represented as

$$\mathbf{b} \equiv \begin{pmatrix} b_\xi \\ b_u \\ b_\omega \end{pmatrix} = \begin{pmatrix} [(1-\beta\rho_\xi)\phi_x + \kappa\phi_\pi]\Omega_\xi \\ [\rho_u\phi_x + \gamma(1-\rho_u)\phi_\pi]\Omega_u \\ [\gamma(1-\rho_\omega)(1-\beta\rho_\omega) - \kappa\rho_\omega]\Omega_\omega \end{pmatrix}. \tag{8}$$

Here Ω_s is used just for notational convenience and defined as

$$\Omega_s = \frac{1}{(1-\beta\rho_s)[\phi_x + \gamma(1-\rho_s)] + \kappa(\phi_\pi - \rho_s)}. \tag{9}$$

The model with purely forward-looking behavior as (1) and (2) results in a simple form of dynamics (7), in which the three variables are represented as affine functions of the three shocks. The simple dynamics—which have no state variable other than the three shocks—makes it possible to obtain closed form solutions for term structure of interest rates.

Since Ω_s is positive, b_ξ and b_u are guaranteed to be positive. This means that the monetary policy authority raises the short rate for counteracting inflationary pressure caused by a rise in the IS shock or the cost push shock. The factor loading on the monetary policy shock b_ω is also positive if and only if

$$\frac{\kappa}{\gamma} < \frac{(1-\rho_\omega)(1-\beta\rho_\omega)}{\rho_\omega}. \tag{10}$$

If this inequality fails, a positive monetary policy shock does not raise the short rate. That is, the direct upward effect of the positive policy shock on the short rate is not stronger than the indirect downward effect of the reflections to the reduced output gap and inflation rate. This is because a large slope of the Phillips curve, a large intertemporal elasticity of substitution, or a persistent monetary policy shock enhances the transmission mechanism of the monetary policy too much, and reverses the reacts of the short rate to the policy shock. This paper regards this odd case as unreasonable, and focuses on the case in which (10) holds. Thus $\mathbf{b} > \mathbf{0}$ and the short rate is procyclical to all shocks.

Even with the simple model, all signs of the relationship between the variables and the shocks represented in (7) make sense. A higher IS shock leads to higher output gap and inflation, to which the monetary policy authority responds by raising the short rate. An increase in the cost push shock raises the inflation. For counteracting the inflationary pressure, the monetary policy authority suppresses the output gap by raising the short rate. An increase in the monetary policy shock raises the short rate, which lowers the output gap and the inflation. The dynamics (7), in which the variables are represented as affine

functions of the shocks, is used for considering the term structure of nominal rates in the next section.

3. Nominal Term Premium

This section considers the nominal term premium. First, the signs of the market prices of risk, the sensitivities of the stochastic discount factor (SDF) to innovations, are discussed, since they are key for understanding the term premia. Then, the closed form solution for nominal term structure, which is consistent with the microfoundation of the new Keynesian model introduced in Section 2, is derived and the properties are discussed.

3.1. The Nominal Stochastic Discount Factor

From the microfoundation for the new Keynesian model introduced in Section 2, the fundamental asset pricing equation for bond prices,

$$V_t^{(n)} = E_t[Q_{t+1} V_{t+1}^{(n-1)}] \tag{11}$$

for n = 1, 2,… can be derived as shown in Appendix A. Here $V_t^{(n)}$ is the price of n-period risk-free discount bond, which pays one unit of money at $t+n$, while Q_{t+1} is the SDF. The log of SDF is represented as

$$q_{t+1} = \ln \beta - \gamma \Delta x_{t+1} - \pi_{t+1} + \frac{1}{1-\rho_\xi} \Delta \xi_{t+1}. \tag{12}$$

The fundamental asset pricing equation (11) is used for obtaining the closed form solution for the interest rate term structure. A special case of the equation, in which $n=1$, is a typical Euler condition for the optimal intertemporal allocation of consumption,

$$V_t^{(1)} = E_t[Q_{t+1}], \tag{13}$$

which is used for deriving the IS curve. Thus the term structure and the IS curve should be internally consistent through the common SDF.

Using (4)-(7), the log of SDF (12) can be rewritten as

$$q_{t+1} = -i_t + \sum_{s=\{\xi,u,\omega\}} \left[-\frac{1}{2}(\nu_s \sigma_s)^2 + \nu_s \sigma_s \varepsilon_{s,t+1} \right], \tag{14}$$

where

$$\mathbf{v} = \begin{pmatrix} \nu_\xi \\ \nu_u \\ \nu_\omega \end{pmatrix} = \begin{pmatrix} [(1-\beta\rho_\xi)\phi_x + \kappa(\phi_\pi - 1)]\Omega_\xi / (1-\rho_\xi) \\ [-\phi_x + \gamma(\phi_\pi - 1)]\Omega_u \\ [\gamma(1-\beta\rho_\omega) + \kappa]\Omega_\omega \end{pmatrix}. \tag{15}$$

The sensitivity of the log of SDF to a unit innovation of shock s, $\nu_s \sigma_s$, is called the price of risk. The sign of ν_s is important for considering the term structure of interest rates. In fact, as will be discussed later, a positive ν_s is necessary for the nominal premium paid for risk of time-variation in shock s to be positive. As can been easily shown by (15), ν_ξ and ν_ω are guaranteed to be positive, but the sign of ν_u depends on the parameter values.

A positive innovation of the IS shock has a positive effect on the SDF directly through the last term of right-hand side of (12), in which the innovation is represented as an increase in ξ_{t+1}. However, through the second and third terms, the innovation has a negative effect on the SDF by raising the output gap and inflation. The positive sign of ν_ξ

is natural since this means that the direct effect of the innovation has more impacts than the indirect effects.

A positive innovation of the cost push shock raises inflation, which results in a lower SDF through the third term in the right-side of (12). On the other hand, through the second term, the SDF is raised since the output gap is suppressed by the monetary policy authority counteracting the inflationary pressure. Thus the sign of the response of SDF depends on which of the second or third term has a larger impact. This may be understood more easily by using the original form of the SDF:

$$Q_{t+1} = \beta \frac{\Psi_{t+1} C_{t+1}^{-\gamma} / P_{t+1}}{\Psi_t C_t^{-\gamma} / P_t} \tag{16}$$

The SDF Q_{t+1} is proportional to the marginal utility of income at $t+1$, $\Psi_{t+1} C_{t+1}^{-\gamma} / P_{t+1}$. An increase in cost push shock (u_{t+1}) raises the price level P_{t+1}, which lowers the marginal utility of income. On the other hand, the monetary policy authority responds to the inflationary pressure by suppressing the consumption level C_{t+1}, which raises the marginal utility of income. Thus if the former direct effect of the cost push shock on the marginal utility of income is larger than the later indirect effect through the monetary policy, the sign of the response of SDF is negative. In fact, ν_u is negative according to the calibration basically following Woodford (2003) as summarized in Table 1[1].

[1] The response coefficient to output gap, 0.125, is equivalent to the Taylor's (1993) estimate, 0.5, since interest rates and inflation rate are annualized in Taylor (1993) while not in the current paper. All volatility parameters are calibrated to the same value, 1%, for easily identifying the influence of the other parameters. This does not matter for considering relative sizes of nominal and real term premia for each shock, since both of them are proportional to the square of volatility of the innovation. On the other hand, this matters for comparing premia for different shocks. In fact, the volatility of innovation of IS shock, which reflects both preference shock and technology shock, is calibrated larger than those of the other shocks in the literature. Thus the relative size of premia for the IS shock is plausibly underestimated in our calibration exercises.

It is easier to understand the positive sign of ν_ω. A positive innovation of the monetary policy shock suppresses output gap and inflation to raise the SDF through both the second and third terms in the right-hand side of (12). In sum, it is natural that $\nu_\xi > 0$, $\nu_u < 0$, and $\nu_\omega > 0$ hold, and these signs are assumed for later discussions.

Equation (15) implies that the prices of risk are time-invariant. In addition, as seen in (7), the short rate is represented as an affine function of the three factors that obey AR(1) processes. These properties of the model are mathematically identical with the three-factor Vasicek term structure model, although this model has macroeconomic interpretations in contrast with purely financial models like Vasicek's[2]. This type of model is known to provide a closed form solution for the term structure of interest rates, as will be obtained in the next subsection.

3.2. Nominal Term Premium

This subsection derives the closed form solution for the nominal forward rate $i_t^{(n)} \equiv -\log\{V_t^{(n)} / V_t^{(n-1)}\}$, defined as the continuously compounded rate which is contracted at t to borrow or lend money starting at $t+n-1$, and to be paid back at $t+n$. As shown in Appendix B, this is represented as an affine function of the shocks,

$$i_t^{(n)} = \bar{i}^{(n)} + \sum_{s=\{\xi,u,\omega\}} b_s^{(n)} s_t \,, \tag{17}$$

where

[2] Vasicek's (1997) and Cox, Ingersoll and Ross's (1985) are the most important classic term structure models. It has been emphasized in the literature that Cox et al.'s model is consistent with a general equilibrium model. In fact, so is Vasicek's model as derived here.

$$b_s^{(n)} = b_s \rho_s^{\ n-1}, \tag{18}$$

$$\bar{i}^{(n)} - \bar{i} = \sum_{s=\{\xi,u,\omega\}} \vartheta_s^{(n)} \tag{19}$$

$$\vartheta_s^{(n)} = \left[\nu_s \{ b_s \bar{\rho}_s^{(n)} \} - \frac{1}{2} \{ b_s \bar{\rho}_s^{(n)} \}^2 \right] \sigma_s^{\ 2} \tag{20}$$

$$\bar{\rho}_s^{(n)} = \sum_{j=1}^{n-1} \rho_s^{\ j-1} = \frac{1-\rho_s^{\ n-1}}{1-\rho_s} \tag{21}$$

for $n = 1, 2, \ldots$ Here $\vartheta_s^{(n)}$ and $\bar{\rho}_s^{(n)}$ are defined for convenience for later discussions. From (12) and (14), the steady state value of the short rate can be proved to be represented as

$$\bar{i} = \ln \frac{1}{\beta} + \bar{\pi} - \frac{1}{2} \sum_{s=\{\xi,u,\omega\}} (\nu_s \sigma_s)^2 \ . \tag{22}$$

Since s_t has an unconditional zero mean, $\bar{i}^{(n)}$ can be interpreted as the unconditional mean of the forward rate. The unconditional mean of the term spread $\bar{i}^{(n)} - \bar{i}$ is equal to the mean term premium denoted as $\vartheta^{(n)}$ by definition. In fact, the term premium is time-invariant in the model. Thus the nominal term premium equals $\vartheta^{(n)}$ and can be represented as a sum of three premia $\vartheta_\xi^{(n)}$, $\vartheta_u^{(n)}$, and $\vartheta_\omega^{(n)}$, each of which is paid for risk of time-variation in each shock.

The first term in the bracket in (20), $\nu_s \{ b_s \bar{\rho}_s^{(n)} \}$, corresponds to the conditional covariance between the log of SDF and the log of bond price through the shock *s*. Remember that the sensitivity of the SDF to the innovation is $\nu_s \sigma_s$. On the other hand, according to (17), (18), and (21), the log of bond price is represented as

$$\log\{V_{t+1}^{(n)}\} = -\sum_{\tau=1}^{n} \bar{i}^{(\tau)} - \sum_{s=\{\xi,u,\omega\}} \{b_s \bar{\rho}_s^{(n)}\} s_{t+1}, \tag{23}$$

and the sensitivity of the log of bond price to a unit innovation of shock s equals $-b_s \bar{\rho}_s^{(n)} \sigma_s$. Thus the covariance through the shock s equals $-\nu_s \{b_s \bar{\rho}_s^{(n)}\} \sigma_s{}^2$. A negative covariance, which means the bond prices are higher when the marginal utility of income is lower, is unfavorable for bondholders. Thus the premium is higher as $\nu_s \{b_s \bar{\rho}_s^{(n)}\}$ is larger. The sign of this term equals that of ν_s, because $b_s \bar{\rho}_s^{(n)}$ is positive. Therefore this term is positive for the IS shock and the monetary policy shock, while negative for the cost push shock. The second term in the bracket of (20), $-\{b_s \bar{\rho}_s^{(n)}\}^2 / 2$, is the Jensen term, which always contributes to lower the term premium. The first term should dominate the second term for a positive premium. Thus since the first term for the cost push shock is negative, so is the corresponding premium: $\vartheta_u^{(n)} < 0$.

The negative premium paid for the cost push shock may be counterintuitive, since this implies that a higher volatility of cost push shock, which is associated with a higher volatility of inflation, results in a lower term premium. This can be understood intuitively as follows. A positive cost push shock raises the inflation rates, which results in lower bond prices since monetary policy authority raises interest rates for counteracting the inflationary pressure. Simultaneously, the higher inflation also lowers the benefit from a unit of money, which could be obtained by selling bonds, and causes bondholders to be less concerned about the nominal bond prices. Accordingly the bond prices are lower when they less concern bondholders. This correlation is favorable for bondholders, and a negative premium is accepted.

In equation (20), which represents the premium for shock s, only $\bar{\rho}_s^{(n)}$ depends on the maturity n. $\bar{\rho}_s^{(n)}$ is a monotonically increasing function of n, and takes 0 when $n=1$ and converges to $\bar{\rho}_s^{(\infty)} \equiv 1/(1-\rho_s)$ as $n \to \infty$. The premium paid for the shock *s*, $\vartheta_s^{(n)}$, is expressed as a quadratic function of $\bar{\rho}_s^{(n)}$, and this relationship is illustrated in Figure 2. Panel (a) displays the case where $\nu_s > 0$, which holds for the IS shock and monetary policy shock. In the case, the premium curve may be hump-shaped and peak when $\bar{\rho}_s^{(n)} = \nu_s / b_s$. This is true for the IS shock since $\bar{\rho}_\xi^{(\infty)} > \nu_\xi / b_\xi$ can be proved using (8) and (15). However, this is not for the monetary policy shock since $\bar{\rho}_\omega^{(\infty)} < \nu_\omega / b_\omega$. Thus the premium for the monetary policy shock is monotonically increasing with the maturity. Panel (b) illustrates the case where $\nu_s < 0$, which holds for the cost push shock. This panel suggests that the premium is negative and monotonically decreasing. Calibration result shown in Figure 3 confirms these analyses. That is, the nominal premium for the IS shock is hump-shaped, that for cost push shock is negative and monotonically decreasing, and that for the monetary policy shock is positive and monotonically increasing. Remember that the premium curve observed in Figure 1 is hump-shaped. This may suggest that the IS shock plays an important role for determining the premium curve[3].

[3] The model predicts that the premium curve for the IS shock peaks at a maturity around three years, which is shorter than 11-13 years observed in data. An important reason for this result is that the simple model lacks mechanisms of endogenous persistence such as habit formation, which help the shocks have more persistent effects on short rates and larger effects on longer term rates. In fact, exogenous persistence can play a similar role for extending the peak of premium curve. If the parameter of autocorrelation of IS shock equals 0.99 instead of 0.95, the model-implied premium curve for the IS shock peaks at a maturity around 12 years.

4. Real Term Premia and Inflation Risk Premia

This section considers real term premia rates and inflation risk premia. First, the fundamental asset pricing equation for real bond prices is reviewed. Then, using closed form solutions, the properties of the real term premia and inflation risk premia are discussed parallel to the discussion of nominal premia in Section 3.

4.1. The Fundamental Asset Pricing Equation for Real Bond Prices

The fundamental asset pricing equation for nominal bond prices (11) leads to

$$V_t^{(n)} = E_t[Q_{t+1}Q_{t+2}\cdots Q_{t+n}], \tag{24}$$

which means that the price of n-period discount bond, whose only payment is one at n periods ahead, is the expectation of the payment discounted by the SDFs. Similarly, the price of n-period real discount bond, which pays P_{t+n}/P_t at the maturity, is represented as

$$\tilde{V}_t^{(n)} = E_t[Q_{t+1}Q_{t+2}\cdots Q_{t+n}P_{t+n}/P_t]. \tag{25}$$

This can be rewritten as

$$\begin{aligned}\tilde{V}_t^{(n)} &= E_t[Q_{t+1}P_{t+1}/P_t E_{t+1}[Q_{t+2}\cdots Q_{t+n}P_{t+n}/P_{t+1}]] \\ &= E_t[\tilde{Q}_{t+1}\tilde{V}_{t+1}^{(n-1)}]\end{aligned} \tag{26}$$

where the log of $\tilde{Q}_{t+1}$ is defined as

$$\begin{aligned}\tilde{q}_{t+1} &= q_{t+1} + \pi_{t+1} \\ &= \ln\beta - \gamma\Delta x_{t+1} + \frac{1}{1-\rho_\xi}\Delta\xi_{t+1}\end{aligned} \tag{27}$$

Equation (26) is the fundamental asset pricing equation for real bond prices, while $\tilde{Q}_{t+1}$ can be called the real SDF.

Using a similar method to that used for the nominal SDF, the log of the real SDF can be rewritten as

$$\tilde{q}_{t+1} = -r_t + \sum_{s=\{\xi,u,\omega\}} \left[-\frac{1}{2}(\tilde{\nu}_s \sigma_s)^2 + \tilde{\nu}_s \sigma_s \varepsilon_{s,t+1} \right], \tag{28}$$

where

$$\tilde{\mathbf{v}} = \begin{pmatrix} \tilde{\nu}_\xi \\ \tilde{\nu}_u \\ \tilde{\nu}_\omega \end{pmatrix} = \begin{pmatrix} [(1-\beta\rho_\xi)\phi_x + \kappa(\phi_\pi - \rho_\xi)]\Omega_\xi / (1-\rho_\xi) \\ \gamma(\phi_\pi - \rho_u)\Omega_u \\ \gamma(1-\beta\rho_\omega)\Omega_\omega \end{pmatrix}, \tag{29}$$

$$r_t = \bar{r} + \sum_{s=\{\xi,u,\omega\}} \tilde{b}_s s_t \,, \tag{30}$$

$$\bar{r} = \ln\frac{1}{\beta} - \frac{1}{2}\sum_{s=\{\xi,u,\omega\}} (\tilde{\nu}_s \sigma_s)^2 \,, \tag{31}$$

$$\tilde{b}_s = (1-\rho_s)\tilde{\nu}_s \,. \tag{32}$$

Here r_t is the real short rate which satisfies $r_t = -\ln \tilde{V}_t^{(1)}$ while $\bar{r}$ is its steady state value. Two properties are worth noting.

First, $\tilde{b}_s > 0$ and $\tilde{\nu}_s > 0$ are guaranteed for all shocks. In particular, $\tilde{\nu}_u > 0$ should be remarked since the sign of nominal counterpart ν_u depends on the parameter values. Remember that a positive innovation of the cost push shock raises inflation to reduce the nominal SDF, but lowers output gap to raise the SDF. Thus the sign of the response of nominal SDF to the innovation depends on which of inflation or output gap has

a larger impact. In the real term, however, the SDF does not depend on the inflation rate, and so the positive innovation raises the SDF just through a lowered output gap.

Second, $\tilde{b}_{\xi} < b_{\xi}$, $\tilde{b}_{u} < b_{u}$, and $\tilde{b}_{\omega} > b_{\omega}$ can be proved. Since an increase in the IS shock or the cost push shock raises expected inflation rate, the nominal rate responds to the shocks more strongly than the real one does. On the other hand, since the monetary policy shock suppresses the expected inflation rate, the nominal rate responds less strongly. The relative seizes of factor loadings are an important property for determining relative sizes of nominal and real term premia as will be discussed in the next section.

4.2. Real Term Premia

The term structure of real forward rates can be obtained with a similar method to that used for the nominal term structure. The real forward rate $r_t^{(n)}$ can be represented as

$$r_t^{(n)} = \bar{r}^{(n)} + \sum_{s=\{\xi,u,\omega\}} \tilde{b}_s^{(n)} s_t \,, \tag{33}$$

where

$$\tilde{b}_s^{(n)} = \tilde{b}_s \rho_s^{\,n-1} \,, \tag{34}$$

$$\tilde{\vartheta}^{(n)} = \bar{r}^{(n)} - \bar{r} = \sum_{s=\{\xi,u,\omega\}} \tilde{\vartheta}_s^{(n)} \,. \tag{35}$$

$$\begin{aligned} \tilde{\vartheta}_s^{(n)} &= \left[\tilde{\nu}_s \{\tilde{b}_s \bar{\rho}_s^{(n)}\} - \frac{1}{2}\{\tilde{b}_s \bar{\rho}_s^{(n)}\}^2 \right] \sigma_s^{\,2} \\ &= \frac{\tilde{\nu}_s^{\,2}}{2}[1-\rho_s^{\,2(n-1)}]\sigma_s^{\,2} \end{aligned} \tag{36}$$

Here $\tilde{\vartheta}^{(n)}$ is the real term premium. Since $\tilde{b}_s$ and $\tilde{\nu}_s$ are related in a simple form (32), the real premium paid for shock *s*, $\tilde{\vartheta}_s^{(n)}$, can be represented with a simpler form than that of the nominal one. In contrast with nominal ones, (36) suggests that all real premia are guaranteed to be positive and monotonically increasing with the maturity.

It is worth considering whether higher factor loadings of nominal and real rates result in a higher absolute value of premium[4]. Using (20), the absolute value of the nominal premium paid for shock s is represented as

$$\left|\vartheta_s^{(n)}\right| = b_s \left| b_s - \frac{2\nu_s}{\bar{\rho}_s^{(n)}} \right| \frac{\{\bar{\rho}_s^{(n)}\sigma_s\}^2}{2} \tag{37}$$

for $n \geq 2$. This suggests that a higher factor loading b_s tends to result in a higher absolute value of premium $|\vartheta_s^{(n)}|$, although this positive correlation possibly fails depending on ν_s and $\bar{\rho}_s^{(n)}$. This discussion is applicable for the real premium as well, since (36) is identical with (20) if $\tilde{b}_s$ and $\tilde{\nu}_s$ are substituted for b_s and ν_s. In fact, using (32), the absolute value of real premium can be represented as

$$\left|\tilde{\vartheta}_s^{(n)}\right| = \tilde{b}_s \left| 1 - \frac{2}{(1-\rho)\bar{\rho}_s^{(n)}} \right| \frac{\{\bar{\rho}_s^{(n)}\sigma_s\}^2}{2}, \tag{38}$$

which suggests that the positive correlation between the factor loading and the absolute value of premium is guaranteed.

Figures 3 reports real term premia other than nominal ones. The panels confirm that the real premium curves have positive slopes. Not only the real premium but also the

[4] Note that higher response coefficients of short rate to output gap and inflation, ϕ_x and ϕ_π, do not necessarily lead to higher factor loadings, since an aggressive monetary policy rule reduces the volatilities of output gap and inflation, which may cause lower factor loadings.

inflation risk premium, which is defined as the nominal premium minus the real premium, are discussed as follows. Panel (a) shows that the inflation risk premium curve due to the IS shock is hump-shaped, because the nominal premium curve is hump-shaped while the real one is monotonically increasing. Panels (b) shows that the absolute value of real premium for the cost push shock is smaller than that of nominal one. An important reason for this result is $\tilde{b}_u < b_u$: the real rate does not respond to the cost push shock as well as the nominal rate does. Note that the inflation risk premium due to the cost push shock is negative, since this equals the negative nominal premium minus the positive real premium. Thus a higher volatility of cost push shock, which is accompanied by a higher volatility of inflation, results in a lower inflation risk premium. Panel (c) confirms that both nominal and real premia paid for the monetary policy shock are positive. The panel also shows that the real one is larger than the nominal one, and so the inflation risk premium is negative. This is consistent with the fact that the real rate responds to the shock more strongly than the nominal rate does.

In sum, the inflation risk premia for cost push shock and monetary policy shock are negative, because the nominal premia are smaller than the real ones. Thus only IS shock can explain the positive inflation risk premium, which is reported in the empirical studies[5].

5. Conclusion

[5] See, for instance, Ang and Beckert (2006).

Why is the average nominal forward rate curve hump-shaped? Which risks are paid for with the term premium? To answer these questions, a typical new Keynesian model with three shocks, the IS shock, cost push shock, and monetary policy shock, is applied. The results are summarized in Table 2 and as follows.

The IS shock may explain the hump-shaped term structure of the nominal term premium. The inflation risk premium curve corresponding to this shock is hump-shaped as well, since the real term premium is guaranteed to monotonically increase with the maturity. The hump-shaped nominal term premium and inflation risk premium are consistent with data and empirical results in the literature.

The nominal premium paid for the cost push shock is negative under plausible conditions. This result is perhaps counterintuitive, since a higher volatility of cost push shock, which is accompanied by a higher volatility of inflation, results in a lower term premium. A positive cost push shock raises the expected inflation rates, which results in lower long-term bond prices. Simultaneously, the higher current inflation also lowers the marginal utility of income, benefit from a unit of money obtained by selling bonds, and causes bondholders to be less concerned about the nominal bond prices. Accordingly the bond prices are lower when they less concern bondholders. This correlation is favorable for bondholders, and a negative premium is accepted. The inflation risk premium due to the cost push shock is negative, because the nominal premium is negative while the real one is positive.

Both the nominal and real premia paid for the monetary policy shock are positive and higher for longer maturities. On the other hand, the inflation risk premium due to the shock is negative, because the real premium is larger than the nominal one, which reflects

that a positive monetary policy shock suppresses the expected inflation rate and so the nominal rate responses to the shock less strongly.

Appendix A

This appendix reviews the microfoundation of the new Keynesian model used in the text. In particular, it is focused how the IS curve (1) is derived from the microfoundation, since the internal consistency between the IS curve and the fundamental asset pricing equation is key for analysis. The model is typical one in the new Keynesian literature, although long term bonds are added in the budged constraint for discussing the term structure of interest rates.

A.1. Household

The preference of the representative household is defined over consumption goods, real money balances, and labor hours. The composite consumption good is defined as

$$C_t = \left[\int_0^1 c_{jt}^{(\theta_t - 1)/\theta_t} dj \right]^{\theta_t/(\theta_t - 1)} \tag{A.1}$$

where c_{jt} is the individual good produced by firm j and θ_t is the price elasticity of demand for the individual good. Here θ_t is an exogenous disturbance, which is the source of the cost push shock. P_t is the corresponding price index of these goods defined as

$$P_t = \left[\int_0^1 p_{jt}^{1-\theta_t} dj \right]^{1/(1-\theta_t)} \tag{A.2}$$

where p_{jt} is the price of individual good produced by firm j. In this setting, the demand for good j can be written as

$$c_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\theta_t} C_t . \tag{A.3}$$

The household supplies labor hours h_{jt}, which are used for producing the individual good c_{jt}.

Households maximize an expected present discount value of utility,

$$E_0 \sum_{t=0}^{\infty} \beta^t \Psi_t \left[\frac{1}{1-\gamma} C_t^{1-\gamma} + \frac{\varpi}{1-\zeta} \left(\frac{M_t}{P_t} \right)^{1-\zeta} - \int_0^1 \frac{\chi}{1+\eta} h_{jt}^{1+\eta} dj \right], \qquad \text{(A.4)}$$

where $0 < \beta < 1$ and all of γ, ζ, η, ϖ and χ are positive. Here Ψ_t is the exogenous preference shock, which is a source of the IS shock. M_t denotes nominal money balances.

The household's budget constraint is

$$P_t C_t + M_t + \sum_{n=1}^{\infty} V_t^{(n)} B_t^{(n)} = \int_0^1 w_{jt} h_{jt} dj + \int_0^1 \Pi_{jt} dj + M_{t-1} + T_t + \sum_{n=1}^{\infty} V_t^{(n-1)} B_{t-1}^{(n)} . \quad \text{(A.5)}$$

Here w_{jt} and Π_{jt} are the nominal wage per hour and the household's share in the profits of firm j, respectively. The aggregate real output Y_t is defined to satisfy

$$P_t Y_t = \int_0^1 w_{jt} h_{jt} dj + \int_0^1 \Pi_{jt} dj . \qquad \text{(A.6)}$$

$V_t^{(n)}$ is the price of *n*-period risk-free discount bond which pays one unit of money at $t+n$, while $B_t^{(n)}$ is the number of the *n*-period bond held by the household at the end of period *t*. T_t is a nominal lump sum transfer that the household receives from the monetary policy authority in the form of cash. This setting of households' maximization problem is typical in the new Keynesian literature, although the long term bonds are added to the budget constraint for considering the term structure of interest rates.

In the household's decision problem, consumption, labor supply, money, and bond holdings are chosen to maximize (A.4) subject to (A.5). One of the first order conditions to the problem is the fundamental asset pricing equation for bond prices (11),

$$V_t^{(n)} = E_t[Q_{t+1} V_{t+1}^{(n-1)}],$$

where Q_{t+1} is the SDF (16).

A.2. The Monetary Policy Authority

The monetary policy authority controls the short rate by obeying a Taylor rule (3). In addition, the policy authority is assumed to just pass out the seigniorage to the representative household as a transfer,

$$T_t = M_t - M_{t-1}. \tag{A.7}$$

Since the bonds are in zero net supply in equilibrium, the budget constraint (A.5) results in $C_t = Y_t$. That is, aggregate consumption and output need not be distinguished from one another in the model.

A.3. Firms

An economy with a continuum of identical firms is considered. For simplicity, variation in capital is ignored so that output of individual good c_{jt} is a function of exogenous productivity A_t and labor input h_{jt} as

$$c_{jt} = A_t h_{jt}. \tag{A.8}$$

Firms maximize their expected profits

$$E_0 \sum_{t=0}^{\infty} \left[\prod_{\tau=1}^{t} Q_\tau (p_{jt} c_{jt} - w_{jt} h_{jt}) \right] \tag{A.9}$$

subject to three constraints. The first is their production function (A.8) and the second is the demand curve given by (A.3). The third constraint is the friction that each period, only a fraction $1-\alpha$ of all firms are able to adjust their prices as a Calvo's (1983) sticky price model.

A.4. Efficient Level of Output and the Output Gap

The efficient level of output is defined as the output chosen by a social planner who can overcome the friction to adjust the prices. Substituting (A.8) into (A.1) results in

$$C_t = A_t \left[\int_0^1 h_{jt}^{(\theta_t - 1)/\theta_t} dj \right]^{\theta_t/(\theta_t - 1)}. \tag{A.10}$$

The social planner maximizes the household's utility (A.4) subject to (A.10). The efficient level of output Y_t^* is defined as the consumption level that satisfies the first-order conditions to the problem:

$$Y_t^* = \chi^{-\frac{1}{\eta+\gamma}} A_t^{\frac{\eta+1}{\eta+\gamma}}. \tag{A.11}$$

Thus the efficient level of output depends positively on the productivity and can be interpreted as a technology shock.

In fact, under friction, in which all firms are not able to adjust their prices at each period, the actual output can deviate from the efficient level of output. The output gap x_t is defined as the log difference between these outputs:

$$x_t = y_t - y_t^*, \tag{A.12}$$

where $y_t = \ln Y_t$ and $y_t^* = \ln Y_t^*$. With these settings, a log-linearization of the first order condition of the firm's maximization problem around a zero average inflation approximately results in the new Keynesian Phillips curve (2) where $\kappa = (1-\alpha)(1-\alpha\beta)/\alpha$. Here the inflation rate is defined as $\pi_t = \ln P_t / P_{t-1}$.

A.5. The Forward-looking IS Curve

Using (A.12), the log of the SDF can be written as (12),

$$q_{t+1} = \ln\beta - \gamma\Delta x_{t+1} - \pi_{t+1} + \frac{1}{1-\rho_\xi}\Delta\xi_{t+1}.$$

where the IS shock ξ_t is defined as the deviation of $(1-\rho_\xi)(\psi_t - \gamma y_t^*)$ from its steady state. Thus a positive IS shock can be interpreted as a positive preference shock or a negative technology shock. Here just for simplicity, the deterministic trend of y_t^* is assumed to equal zero. Since $E_t\Delta\hat{\xi}_{t+1} = -(1-\rho_\xi)\hat{\xi}_t$, log-linearizing the Euler condition (13) around the steady state results in the IS curve (1).

Appendix B

By substituting (14) and (17) into (11),

$$\begin{aligned}&\exp\left\{-\sum_{\tau=1}^{n}\left(\bar{i}^{(\tau)} + \sum_{s=\{\xi,u,\omega\}} b_s^{(\tau)} s_t\right)\right\}\\ &= E_t\left[\exp\left\{-\bar{i} + \sum_{s=\{\xi,u,\omega\}}\left(-b_s s_t - \frac{1}{2}(\nu_s\sigma_s)^2 + \nu_s\sigma_s\varepsilon_{s,t+1}\right) - \sum_{\tau=1}^{n-1}\left(\bar{i}^{(\tau)} + \sum_{s=\{\xi,u,\omega\}} b_s^{(\tau)} s_{t+1}\right)\right\}\right].\end{aligned} \tag{B.1}$$

Since s_t obeys AR(1) represented in (4)-(6),

$$\exp\left\{-\bar{i}^{(n)} - \sum_{s=\{\xi,u,\omega\}} \sum_{\tau=1}^{n} b_s^{(\tau)} s_t\right\}$$
$$= E_t\left[\exp\left\{-\bar{i} - \frac{1}{2}\sum_{s=\{\xi,u,\omega\}} (\nu_s \sigma_s)^2 - \sum_{s=\{\xi,u,\omega\}} \left(b_s + \rho_s \sum_{\tau=1}^{n-1} b_s^{(\tau)}\right) s_t + \sum_{s=\{\xi,u,\omega\}} \left(\nu_s - \sum_{\tau=1}^{n-1} b_s^{(\tau)}\right) \sigma_s \varepsilon_{s,t+1}\right]. \tag{B.2}$$

It follows, using $\varepsilon_{l,t} \sim N(0,1)$ i.i.d., that

$$-\bar{i}^{(n)} - \sum_{s=\{\xi,u,\omega\}} \sum_{\tau=1}^{n} b_s^{(\tau)} s_t$$
$$= -\bar{i} - \frac{1}{2}\sum_{s=\{\xi,u,\omega\}} (\nu_s \sigma_s)^2 - \sum_{s=\{\xi,u,\omega\}} \left(b_s + \rho_s \sum_{\tau=1}^{n-1} b_s^{(\tau)}\right) s_t + \frac{1}{2}\sum_{s=\{\xi,u,\omega\}} \left(\nu_s - \sum_{\tau=1}^{n-1} b_s^{(\tau)}\right)^2 \sigma_s^{\,2}. \tag{B.3}$$

Since (B.3) holds for any outcome of ξ_t, u_t and ω_t,

$$\sum_{\tau=1}^{n} b_s^{(\tau)} = b_s + \rho_s \sum_{\tau=1}^{n-1} b_s^{(\tau)}, \tag{B.4}$$

$$\bar{i}^{(n)} = \bar{i} + \left[\sum_{s=\{\xi,u,\omega\}} \left\{\nu_s \sum_{\tau=1}^{n-1} b_s^{(\tau)} - \frac{1}{2}\left(\sum_{\tau=1}^{n-1} b_s^{(\tau)}\right)^2\right\}\right] \sigma_s^{\,2}. \tag{B.5}$$

Solving (B.4) results in (18), and substituting this into (B.5) leads to (19).

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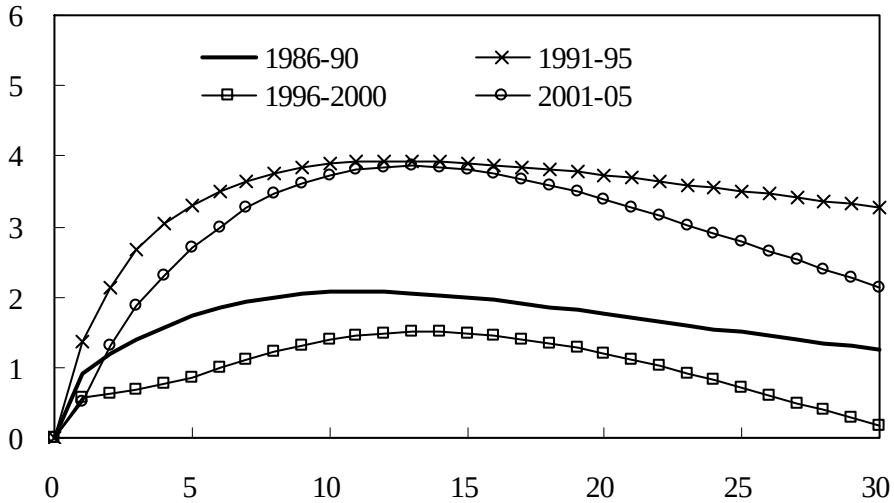
Table 1: The Calibrated Parameters

β	0.99	ρ_{ξ}	0.95
γ	0.16	ρ_{u}	0.80
κ	0.024	ρ_{ω}	0.65
ϕ_{x}	0.125	σ_{ξ}	0.01
ϕ_{π}	1.5	σ_{u}	0.01
		σ_{ω}	0.01

Table 2: Summarized Results

	Nominal term premia	Real term premia	Inflation risk premia
IS shock	Depends / hump-shaped	Positive / increasing	Depends / hump-shaped
Cost push shock	Negative / decreasing	Positive / increasing	Negative / decreasing
Monetary policy shock	Positive / increasing	Positive / increasing	Negative / decreasing

Figure 1: Average Premium Curves



Note: This figure reports five-year average term premia of forward rates (percent per annual), defined as instantaneous forward rates minus the instantaneous zero-coupon yield. Data are provided by Gurkaynak, Sack, and Wright (2006). The horizontal axis corresponds to the maturity (years).

Figure 2: Illustrations of the Relationship between Parameter Values and Term Premia

(a) $\nu_s > 0$ (b) $\nu_s < 0$

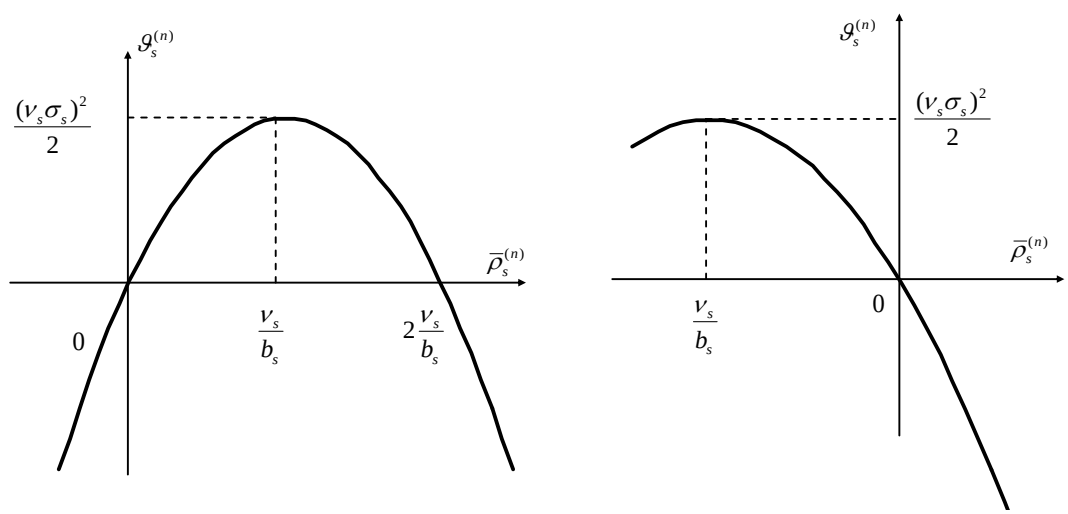
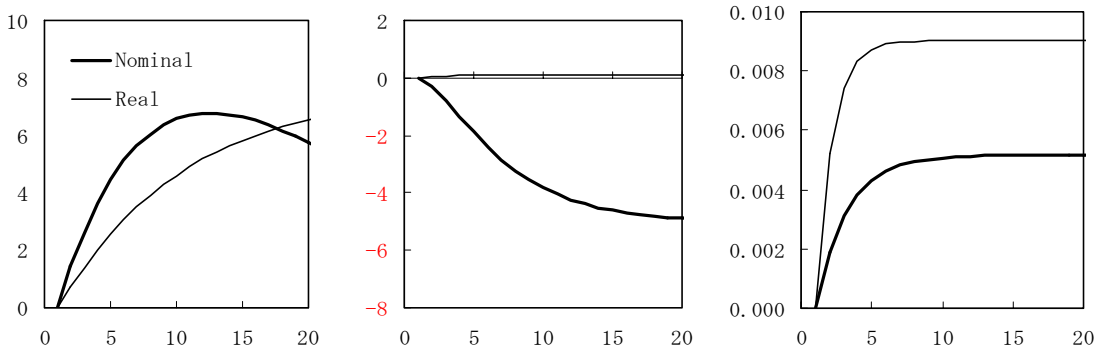


Figure 3: Decompositions of Nominal and Real Term Premia

(a) IS shock (b) Cost push shock (c) Monetary policy shock



Note: Each panel reports the model-implied nominal and real term premia (percent per annual) paid for risk of time-variation in the corresponding shock. The horizontal axes correspond to the maturity (quarters).